SDP for Algebra, Combinatorics & Geometry **MathCoRe**

Lorenzo Baldi
Sebastian Debus

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Symmetric polynomials

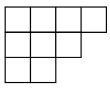
Exercise 1

Let $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$ be symmetric real polynomials of degrees $d_1, \dots, d_m$. Show that the real variety $V(f_1, \dots, f_m) = \{z \in \mathbb{R}^n : f_1(z) = \dots = f_m(z) = 0\}$ is non-empty if and only if it contains a point $z \in \mathbb{R}^n$ with at most $\max\{2, d_1, \dots, d_m\}$ distinct coordinates.

Hint: Can $f_1 = 0, \dots, f_m = 0$ be defined by only one polynomial equations with higher degree? Note that this is different over $\mathbb{C}$.

Exercise 2

Let $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$ be a partition (a sequence of decreasing nonnegative integers that sum up to n). We define a *Young diagram* of *shape* λ as the ordered sequences of boxes with λ_1 boxes in the first row, λ_2 in the second row, ... For instance,



is a diagram of shape $(4, 3, 2)$. A *Young tableau* of shape $\lambda \vdash n$ is a filling of a diagram of shape λ with all the integers $1, 2, \dots, n$. For instance,

9	3	6	4
2	1	8	
5	7		

is a Young tableau for the partition $(4, 3, 2)$. The *Specht polynomial* $\mathrm{sp}_T \in \mathbb{R}[x_1, \dots, x_n]$ associated with a tableau T is the product over all $x_i - x_j$ where $i \neq j$ are contained in the same column of the tableau T and i is written above j. For instance,

$$(x_9 - x_2)(x_9 - x_5)(x_2 - x_5)(x_3 - x_1)(x_3 - x_7)(x_1 - x_7)(x_6 - x_8).$$

A S_n-*orbit type* is of the form $(a_1, a_1, \ldots, a_1, a_2, \ldots, a_2, a_3, \ldots, a_l) \in \mathbb{R}^n$ for pairwise different coordinates $a_i \in \mathbb{R}$ and only unique up to permutation.

1. List all S_n orbit types in $\mathbb{R}^3$.
2. Determine all Young tableaux of shape $(2,1)$ and characterize the real variety

$$V_{(2,1)} = \{z \in \mathbb{R}^3 : \mathrm{sp}_T(z) = 0, \text{ for all tableaux } T \text{ of shape } (2,1)\}.$$

   Can you characterize $V_{(2,1)}$ in terms of orbit types?

3. Characterize $V_{(1,1,1)}$ in terms of orbit types.
4. Can you formulate some possible relations between orbit types and varieties V_λ?

Exercise 3

Recall that $S_n = \{\sigma : [n] \to [n] \mid \sigma \text{ is bijective}\}$ is the symmetric group with respect to the letters $[n]$. Consider the hyperoctahedral group $B_n = \{\pm 1\}^n \ltimes S_n$. An element (τ, σ) with $\tau \in \{\pm 1\}^n$ and $\sigma \in S_n$ acts on $(z_1, \ldots, z_n) \in \mathbb{R}^n$ via

$$(\tau, \sigma) \cdot (z_1, \ldots, z_n) = (\tau_{\sigma(1)} z_{\sigma(1)}, \ldots, \tau_{\sigma(n)} z_{\sigma(n)})$$

1. Determine the points in $\mathbb{R}^n$ that are fixed under all elements of B_n.
2. Try to define and encode an orbit type of an element $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ with respect to the hyperoctahedral group.

Properties of the sums of squares and moment hierarchies

Exercise 4

Consider the Motzkin polynomial $f(x, y) = 1 - 3x^2y^2 + x^2y^4 + x^4y^2$. Consider the unconstrained optimization problem, i.e. $\min f(\mathbf{x}) \colon \mathbf{x} \in \mathbb{R}^2$ (we are then considering the quadratic module $M(1) = \Sigma^2$). Recall that $f^* = 0$. Prove that $f^*_{\mathrm{sos},r} = f^*_{\mathrm{mom},r} = -\infty$ for all r.

Exercise 5

Show that there exists closed convex cones C_1, C_2 whose Minkowski sum is not closesd.

Hint: Consider $C_1 = \Sigma^2_{2,1}$ *and* $C_2 = -\mathbb{R}_{\geq 0}x^2$*, and then use a previous exercise*

Exercise 6

Show that, given $f = x_1x_2$, $g_1 = -x^2$, $g_2 = 1-x_1$, $g_3 = 1+x_2$, then $f^* = 0$, $f^*_{\text{SoS},1} = -\infty$, $f^*_{\text{SoS},1} = 0$. What about higher order of the hierarchies?

Exercise 7

Assume that $M(\mathbf{g})$ is Archimedean, say $1 - x_1^2 - \cdots - x_n^2 \in M(\mathbf{g})$. Then prove that $T(1 - x_1^2, \ldots, 1 - x_n^2) \subset M(\mathbf{g})$.