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Geometry (Spherical packings)

Exercise 1

We want to show that the infinitely many linear inequalities in Delsarte's linear program can be phrased as finitely many positive semidefinite conditions. Therefore a truncated Delsarte's linear program is a semidefinite program.

Use Fekete's theorem: Let $f \in \mathbb{R}[t]$ be a univariate polynomial. Then $f \geq 0$ on $[-1, 1]$ iff $f = \sigma_1 + (1 - x^2)\sigma_2$ for some sums of squares $\sigma_1, \sigma_2 \in \mathbb{R}[t]$, to prove an equivalent condition for $f \geq 0$ on $[a, b]$ using sums of squares polynomials.

Exercise 2

Can you say something about the relation of n, i.e. the number of points, k, i.e the dimension of the vector space $\mathbb{R}^k$, and $\mathrm{rk}(X)$ for a feasible solution X in the SDP relaxation for the problem of reconstructing locations.

Exercise 3

Recall that the Chebyshev polynomials $(P_k^{(2)} \in \mathbb{R}[t])_{k\in\mathbb{N}}$ are the unique univariate polynomials such that $\deg P_k^{(2)} = k$ and

$$\int_{-1}^{1} \frac{P_k^{(2)} P_l^{(2)}}{\sqrt{1-t^2}} \,\mathrm{d}\, t = 0, \forall k \neq l, \quad P_k^{(2)}(1) = 1.$$

Show that the Chebyshev polynomials $P_k^{(2)} \in \mathbb{R}[t]$ define the following recursion:

$$P_0^{(2)} = 1, P_1^{(2)} = t, P_{k+1}^{(2)} = 2tP_k^{(2)} - P_{k-1}^{(2)}$$

Polynomial Optimization: moment relaxations

Exercise 4

1. Show that, given $L \in \mathbb{R}[\mathbf{x}]^\vee$ (i.e. $L\colon \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ linear), $y_\alpha = L(\mathbf{x}^\alpha)$ and $g \in \mathbb{R}[\mathbf{x}]$, then $L(gf^2) \geq 0$ for all $f \in \mathbb{R}[\mathbf{x}]$ if and only if $\mathbf{M}(g \cdot \mathbf{y}) \succcurlyeq 0$ *Hint: adapt the proof from the case of the moment matrices* $\mathbf{M}(\mathbf{y})$
2. Adapt the statement for *truncated* linear functionals, i.e. $L\colon \mathbb{R}[\mathbf{x}]_{2r} \to \mathbb{R}$, and conclude that the $\mathbf{M}_r(\mathbf{y}) \succcurlyeq 0, \mathbf{M}_{k_1}(g_1 \cdot \mathbf{y}) \succcurlyeq 0, \dots \mathbf{M}(g_m \cdot \mathbf{y}) \succcurlyeq 0$ if and only if $L(q) \geq 0$ for all $q \in M(\mathbf{g})_{2r}$. *Hint: write* $q = \sigma_0 + \sigma_1 g_1 + \cdots + g_m \sigma_m$ *and* $\sigma_i = \sum_j f_{i,j}^2, \dots$

Exercise 5

- Prove that, if $\mathbf{x}^*$ is a minimizer of f on $S(\mathbf{g})$, then $\int f \, \mathrm{d}\delta_{\mathbf{x}^*} = f^*$.
- Show that $f^* = \inf\{ \int f \, \mathrm{d}\mu \mid \mu \in \mathcal{M}(S(\mathbf{g})),\ \int 1 \, \mathrm{d}\mu = 1 \}$.
- Conclude that $f^*_{\mathrm{mom},r} \leq f^*$ for all $r \in \mathbb{N}$.