SDP for Algebra, Combinatorics & Geometry

Lorenzo Baldi
Sebastian Debus

1. Day 19.06.2023

Exercise 1

1. Let $A, B \in \mathcal{S}^n_{\succeq 0}$ be positive semidefinite. Show that $\langle A, B \rangle \geq 0$.
2. Show that the proper convex cone of positive semidefinite matrices $\mathcal{S}^n_{\succeq 0}$ is self dual, i.e. $A \succeq 0$ if and only if $\langle A, B \rangle \geq 0$ for all $B \in \mathcal{S}^n_{\succeq 0}$.
3. Let $X = \begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_k \end{pmatrix} \in \mathcal{S}^n$ be a block diagonal symmetric matrix. Show that $X \succeq 0$ if and only if $X_1, \ldots, X_k \succeq 0$.

Exercise 2:

Let $X \in \mathcal{S}^n$ be a real symmetric matrix. Show that the following assertions are equivalent:

1. X is positive semidefinite, i.e. $x^T X x \geq 0$ for all $x \in \mathbb{R}^n$.
2. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of X are all nonnegative.
3. There exists a *Cholesky decomposition* of X, i.e. $X = LL^T$ for some matrix $L \in \mathbb{R}^{n \times k}$.
4. There exist vectors $v_1, \ldots, v_n \in \mathbb{R}^k$ such that $X_{ij} = v_i^T v_j$ for all $i, j \in [n]$. (The v_i's are called a *Gram representation* of G).

Exercise 3

1. Consider the primal program

$$d^* = \sup\left\{\langle\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, X\rangle : X \succeq 0, \langle\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, X\rangle = 1, \langle\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, X\rangle = 0\right\}$$

   and its dual

$$p^* = \inf\left\{y_1 : y_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \succeq 0\right\}.$$

   Determine $d^* = p^*$ and show that only of supremum and infimum is attained. Why does that not contradict the strong duality theorem?

2. Consider the primal semidefinite program with data

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   and $b_1 = 0, b_2 = 1$. Show that there is a positive duality gap and $d^* - p^* = 1$.

Exercise 4

1. Show that a local maximum of a semidefinite program is a global maximum.

   Hint: You can use that an optimization problem with a convex feasible region and linear objective function has the property that any local optimum is a global optimum.

2. We define

$$p_* = \inf\{\langle C, X\rangle : X \succeq 0, \langle X, A_1\rangle = b_1, \ldots, \langle X, A_m\rangle = b_m\}$$

   and

$$d_* = \sup\left\{\sum_{j=1}^{m} b_j y_j : y \in \mathbb{R}^m, C - \sum_{j=1}^{m} y_j A_j \succeq 0\right\}.$$

   What can you say about the relation of p_* and d_*? Does strong duality work analogously?

Exercise 5

Try to solve some SDP numerically using NEOS Server with sedumi in sparse SDPA format. *For instance, solve exercise 3 using NEOS Server.*

https://neos-server.org/neos/solvers/sdp:sedumi/SPARSE_SDPA.html

Here is the manual: http://plato.asu.edu/ftp/sdpa_format.txt