Polynomial Optimization, SoS strenghtening

<u>Recall</u> $f \in \mathbb{R}[x]$, $g_1, \dots, g_m \in \mathbb{R}[x]$.

We defined:

- $f^* = \inf \{ f(x) \mid x \in \mathbb{R}^n \}$
  $= \sup \{ \lambda \in \mathbb{R} \mid f(x) - \lambda \geq 0 \text{ on } \mathbb{R}^n \}$
- $f^*_{SoS} = \sup \{ \lambda \in \mathbb{R} \mid f(x) - \lambda \in \Sigma^2 \} \leq f^*$
  $\hookrightarrow$ SDP Problem

We now look at a more general class of optimization problems: Polynomial Optimization Problems

$$S(\underline{g}) := \{ x \in \mathbb{R}^N \mid g_i(x) \quad \forall i = 1, \dots, m \}$$

$$f^* := \inf \{ f(x) \mid x \in S(\underline{g}) \} =$$
$$= \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \geq 0 \text{ on } S(\underline{g}) \}$$

<u>Proof</u>: (Assuming for simplicity $S(\underline{g})$ compact).

By definition, $f - f^* \geq 0$ on $S(\underline{g})$

Moreover, if $x^* \in S(\underline{g})$ is such that $f(x^*) = f^*$, then $f(x^*) - f^* - \varepsilon < 0$ $\forall \varepsilon > 0$. (Ex.)

Goal of polynomial optimization: find (or approximate) f^*.

Denote $P(S(\underline{g})) = \{ f \in \mathbb{R}[X] \mid f(x) \geq 0 \ \forall x \in S(\underline{g}) \}$.

this is a convex cone: $f_1, f_2 \geq 0$ on $S(\underline{g})$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0} \Rightarrow \lambda_1 f_1 + \lambda_2 f_2 \geq 0$ on $S(\underline{g})$

then:

$$f^* = \sup \{ \underbrace{\lambda \in \mathbb{R}}_{\text{linear objective}} \mid \underbrace{f - \lambda \in P(S(\underline{g}))}_{\text{Conic constraint}} \} \to \text{Convex optimization}$$

Problems: $P(S(\underline{g})) \subseteq \mathbb{R}[X]$ is infinite dimensional

- $P(S(\underline{g}))$ is extremely complicated: no effective description.

Goal: find Semidefinite inner approximations of $P(S(\underline{g}))$, which leave in a finite dimensional vector space.

We come back to f^*_{sos}, and we assume $\deg f = 2d$ (if it has odd degree, then $f^* = -\infty$)

Notice that: $f^*_{sos} = \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in \Sigma^2_{2d,n} \}$

$\to$ we can only consider polynomials of $\deg \leq 2d$!

Lasserre-Parrilo SoS hierarchy

Notice that $M(\underline{g}) = \Sigma^2 + \Sigma^2 g_1 + \dots + \Sigma^2 g_m$ is a convex cone, and $M(\underline{g}) \subseteq P(S(\underline{g}))$.

How do we restrict to a finite dimensional vector space? $\to$ we bound the degrees of the SOS.

<u>Definition</u> the quadratic module (generated by $\underline{g}$) truncated at degree r is the convex cone

$$M(\underline{g})_r := \left\{ \sigma_0 + \sigma_1 g_1 + \dots + \sigma_m g_m \mid \sigma_i \in \Sigma^2, \deg \sigma_0 \le r, \ \forall_i \deg \sigma_i g_i \le r \right\} =$$

$$= \Sigma^2_{r,n} + \Sigma^2_{r-\deg g_1, n}\, g_1 + \dots + \Sigma^2_{r-\deg g_m, n}\, g_m .$$

- From the SDP representation of Σ^2, we can get an SDP representation for $M(\underline{g})_r$.
- By definition, $M(\underline{g})_r \subseteq P(S(\underline{g}))$
- <u>Remark</u>: $M(\underline{g})_r$ is, in general, properly contained in $M(\underline{g}) \cap \mathbb{R}[x]_{\le r}$ (Ex.)

then: for every r, we can substitute $P(S(\underline{g}))$ with $M(\underline{g})_{2r}$.

Def the SoS strengthening at order $r \in \mathbb{N}$

of the polynomial optimization problem

min $f(x)$ s.t. $g_1(x) \geq 0, \dots, g_m(x) \geq 0$

is defined as:

$$f^*_{SoS,r} := \sup \{ \lambda \in \mathbb{R} \mid f - \lambda \in M(\underline{g})_r \}.$$

↳ this can be $\neq$ max, because $M(\underline{g})_r$ is not closed in general.

Properties (Ex.)

- $\forall r,\ f^*_{SoS,r} \leq f^*_{SoS,r+1}$
- $\forall r,\ f^*_{SoS,r} \leq f^*$

• this is a general way of finding SDP approx. of general, non-convex, optimization problems.

the SoS hierarchy is the (increasing) sequence $\{f^*_{SoS,r}\}_{r \in \mathbb{N}}$

Question: do we have convergence $f^*_{SoS,r} \nearrow f^*$?

theorem If $M(\underline{g})$ is Archimedean, then

$$\lim_{r \to +\infty} f^*_{SoS,r} = f^*.$$

Proof Recall Putinar's theorem:

if $h > 0$ on $S(\underline{g})$ and $M(\underline{g})$ is Archimedian,

then $h = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_m g_m \in M(\underline{g})$

$(\sigma_i \in \Sigma^2)$ →

→ Now, for all $\varepsilon > 0$ we have (by definition of f^*)

$f - f^* + \varepsilon > 0$ on $S(\underline{g})$ $\overset{\text{Putinar}}{\Longrightarrow}$ there exists

$\sigma_0, \dots, \sigma_m \in \Sigma^2$ s.t.

$$f - f^* + \varepsilon = \sigma_0 + \sigma_1 g_1 + \dots + \sigma_m g_m.$$

Let r be such that $2r \geq \max\{\deg \sigma_0, \deg \sigma_1 g_1, \dots, \deg \sigma_m g_m\}$.

Therefore, if we choose $\lambda = f^* - \varepsilon$, we have

$f - \lambda \in M(\underline{g})_{2r}$. Then, by definition of $f^*_{SOS,r}$,

$f^*_{SOS,r} \geq f^* - \varepsilon$ and:

$$0 \leq f^* - f^*_{SOS,r} \leq f^* - (f^* - \varepsilon) = \varepsilon.$$

In conclusion, for all $\varepsilon > 0$ we can find $r = r(\varepsilon)$ s.t. $f^* - f^*_{SOS,r} \leq \varepsilon$, concluding the proof.

Questions:

- What about the extraction of minimizers?
- What about finite convergence (i.e. $f^* = f^*_{SOS,r}$)?
- What about the dual SDP of $f^*_{SOS,r}$?

→ tomorrow.