MathCore Compact Course on Scientific Computing

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Agenda

1. Some notes on the Laplace Equation
2. Finite Elements
3. From a pde to a Matrix

Tomorrow

- Numerical complexity of finite element simulations
- Solving linear systems
- Multigrid as tool for complexity-reduction

$$\mathcal{O}\left(N^3\right) \rightarrow \mathcal{O}\left(N\right)$$

Laplace equation

- Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional **domain** (usually $d = 2$ or $d = 3$) and $\Gamma = \partial\Omega$ the boundary of Ω.
- Let f be the **right hand side**
$$f : \bar{\Omega} \to \mathbb{R}$$
- Find the **solution** u
$$u : \bar{\Omega} \to \mathbb{R}$$
such that
$$-\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \Gamma$$
- With the **Laplace-Operator**
$$-\Delta := -\sum_{i=1}^{d} \frac{\partial^2}{\partial^2 x_i}$$

Applications

- Heat diffusion, mechanics of a thin membrane (soap bubbles). One essential part in many important differential equations like Navier-Stokes
- The Laplace equation is the prototypical **elliptic pde**

Regularity

- Considering "common functions", Laplace is an operator

$$-\Delta : C^2(\Omega) \to C(\Omega)$$

such that the proper regularity of the problem is

$$f \in C(\Omega), \quad u \in C^2(\Omega) \cap C(\bar{\Omega})$$

- However: usually there exists no such **classical solution**, e.g.

$$-\Delta u = 1 \text{ in } \Omega = (0,1)^2, \quad u = 0 \text{ on } \partial\Omega$$

does not have a **classical solution** $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

PDE's are made by God, the boundary conditions by the Devil, Alan Turing

It is indeed the boundary that makes the difference. For ode boundary value problems, the boundary constraint is finite dimensional. Also, the kernel of the differential operator is finite dimensional. For pde's everything is usually infinite dimensional (e.g. all harmonic functions satisfy $-\Delta\phi = 0$).

1. Assume that $u : C^2(\Omega) \cap C(\bar{\Omega})$ satisfies the Laplace problem

$$-\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega.$$

2. Multiply the equations with a **test-function** ϕ

$$\phi \in V_0 := \{\psi \in C^1(\Omega) \cap C(\bar{\Omega}) \, : \, u = 0 \text{ on } \partial\Omega\} \quad \Rightarrow \quad -\Delta u \cdot \phi = f \cdot \phi$$

3. Integrate over Ω

$$\Rightarrow \quad -\int_\Omega \Delta u \phi \, dx = \int_\Omega f \phi \, dx.$$

4. Apply integration by parts

$$-\int_\Omega \Delta u \phi \, dx = \sum_{i=1}^{d} \int_\Omega -\partial_{ii} u \phi \, dx = \sum_{i=1}^{d} \int_\Omega \partial_i u \partial_i \phi \, dx - \sum_{i=1}^{d} \int_{\partial\Omega} \vec{n}_i \partial_i u \underbrace{\cdot \phi}_{=0} \, dx = \int_\Omega f \phi \, dx.$$

this is

$$\Rightarrow \quad \int_\Omega \nabla u \cdot \nabla \phi \, dx = \int_\Omega f \phi \, dx \quad \forall \phi \in V_0$$

Notation

Introduce L^2-scalar product and L^2-norm

$$(u,v)_\Omega = (u,v) = \int_\Omega uv\,dx, \quad \|u\|_\Omega = \|u\| = (u,u)^{\frac{1}{2}} = \int_\Omega |u|^2\,dx$$

We get

$$-\Delta u = f \text{ in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega \quad \Rightarrow \quad (\nabla u, \nabla\phi) = (f,\phi) \quad \forall \phi \in V_0$$

"Weaker solution" define $u \in V_0$ as solution to

$$u \in C^1(\Omega) \cap C_0(\bar{\Omega}) : \quad (\nabla u, \nabla\phi) = (f,\phi) \quad \forall \phi \in V_0$$

(weaker, as u only requires C^1-regularity, no 2nd derivatives)

- Is the **"weaker solution"** also a **classical solution**?

  $$(\nabla u, \nabla\phi) = (f,\phi) \quad \forall \phi \in V_0 \quad \overset{?}{\Rightarrow} \quad -\Delta u = f, \quad u \in C^2(\Omega) \cap C_0(\bar{\Omega}).$$

  Only, if $u \in C^2(\Omega)$!

- Does such a "weaker solution" exist? Usually no, the problem $-\Delta u = 1$ on the square with $u = 0$ on the boundary still does not have such a solution with $C^1(\Omega)$-regularity.

Lemma 1 *Every "weaker solution"*

$$u \in V_0 : \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in V_0$$

is unique solution to the minimization problem

$$u \in V_0 : \quad E(u) \le E(v) := \frac{1}{2}\|\nabla v\|^2 - (f, v), \quad \|\phi\| = \int_\Omega |\phi|^2 \, dx.$$

Lemma 2 *Let Ω be a domain with a boundary that is locally given as graph of a Lipschitz continuous function. Let $f \in L^2(\Omega)$. Then there exists a unique solution*

$$u \in H_0^1(\Omega)$$

to the minimization problem. Where $H_0^1(\Omega)$ is the Sobolev space of L^2-functions with weak derivative in L^2.

$$E(u) \leq E(v) \quad \forall v \in V_0$$

$$\frac{d}{ds} E(u+sv)\Big|_{s=0} = 0 \qquad \forall v \in V_0$$

$$= \frac{1}{2}\left(\nabla(u+sv), \nabla v\right) + \frac{1}{2}\left(\nabla v, \nabla(u+sv)\right) - (f, v)\Big|_{s=0}$$

$$= \frac{1}{2}(\nabla u, \nabla v) + \frac{1}{2}(\nabla v, \nabla u) - (f, v)$$

$$= (\nabla u, \nabla \psi) - (f, v) \overset{!}{=} 0 \quad \forall v \in V_0$$

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in V_0$$

$$E(u+v) = \frac{1}{2}\|\nabla(u+v)\|^2 - (f, u) - (f, v)$$

$$= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 + \frac{1}{2}(\nabla u, \nabla v) + \frac{1}{2}(\nabla v, \nabla u) - (f, u) - (f, v)$$

$$= E(u) + \frac{1}{2}\|\nabla v\|^2 + \underbrace{(\nabla u, \nabla v) - (f, v)}_{=0}$$

$$\geq E(u)$$

Proof Existence

$$E(v) = \frac{1}{2}\|\nabla v\|^2 - (f, v)$$

① Show $E(v) > -\infty \quad \forall v$

$$E(v) \geq \frac{1}{2}\|\nabla v\|^2 - \|f\| \cdot \|v\| \quad \text{(Poincare)}$$

$$\geq \frac{1}{2}\|\nabla v\|^2 - \|f\| c_p \|\nabla v\|$$

(Young's inequality: $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$)

$$\geq \frac{1}{2}\|\nabla v\|^2 - \frac{c_p^2}{2}\|f\|^2 - \frac{1}{2}\|\nabla v\|^2$$

$$= -\frac{c_p^2}{2}\|f\|^2 > -\infty$$

(2) Sequence $v_n \in V_0$

$$E(v_n) \longrightarrow d := \inf_{r \in V_0} E(r)$$

Question: $v_n \to r$?

(3) Show: v_n Cauchy-seq

$$\|\nabla(v_n - v_m)\|^2 = \underbrace{4E(v_n)}_{\to 4d} + \underbrace{4E(v_m)}_{\to 4d} \underbrace{- 8E\left(\frac{v_n+v_m}{2}\right)}_{\le -8d}$$

$$\underbrace{+ 4(f, v_n) + 4(f, v_m) - 8\left(f, \frac{v_n+v_m}{2}\right)}_{= 0}$$

$\Rightarrow$ v_n Cauchy

$$H_0^1(\Omega) = \{ \varphi : \Omega \to \mathbb{R} \mid \exists \text{ Cauchy-Sequence in } V_0 \text{ with } \|\nabla(v_n - \varphi)\| \to 0 \}$$

Sketch of the proof

... extra slide

What is $H_0^1(\Omega)$

1. The completion of V_0 with respect to $\|\nabla \cdot\|_{L^2(\Omega)}$.

$$H_0^1(\Omega) := \{\phi \in L^2(\Omega) \,:\, \exists \text{ Cauchy sequence } \phi_n \in V_0 \text{ with } \|\nabla(\phi - \phi_n)\| \to 0.\}$$

2. The space of L^2-functions with first weak derivative in L^2 and **trace zero**:

$$H_0^1(\Omega) = \{\phi \in L^2(\Omega) \,:\, \phi = 0 \text{ on } \Omega,\ \exists w_i \in L^2(\Omega) \,:\, -(\phi, \partial_i \psi) = (w_i, \psi)\ \forall \psi \in C_0^\infty(\Omega)\}.$$

$H_0^1(\Omega)$ functions are

- Continuous in 1d

$$H_0^1([0,T]) \subset C([0,T])$$

- Not necessarily continuous in 2d, 3d. Example

$$u_n(x,y) = \log\left(\log\left(\frac{1}{\sqrt{x^2+y^2+\frac{1}{n}}}\right)+1\right) \text{ on } B_1(0), \quad u_n \in C^1(B_1(0)) \text{ but } \lim_{n\to\infty} u_n(0,0) = \infty.$$

$H_0^1(\Omega)$ is a

- A Hilbert space (complete with scalar product)
- It is called a *Sobolev space*

Weak solution

Let $f \in L^2(\Omega)$. Find

$$u \in H_0^1(\Omega): \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in H_0^1(\Omega)$$

Lemma 3 *If a* ***weak solution*** $u \in H_0^1(\Omega)$ *has the additional regularity*

$$u \in C^2(\Omega) \cap C(\bar{\Omega})$$

then, u is a ***classical solution*** *(also called* ***strong solution****).*

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- The weak solution is defined as

$$u \in V := H_0^1(\Omega) \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in V.$$

- This problem has *infinite dimension* as the space $H_0^1(\Omega)$ does not have a finite basis

Galerkin method

- Choose a finite dimensional subspace $V_h \subset V$ and restrict the problem

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h.$$

- This problem has finite dimension and now fits to the computer

Finite elements are

- A special choice of a function space V_h with a basis

$$V_h = \langle \phi_h^1, \dots, \phi_h^N \rangle$$

such that very few basis functions overlap

$$\mathrm{supp}(\phi_h^i) \cap \mathrm{supp}(\phi_h^j) \neq 0 \text{ only for few } i \neq j$$

- We call a Galerkin approach (H^1-)**conforming** if

$$V_h \subset V = H_0^1(\Omega)$$

- Then it holds

$$\begin{aligned}(\nabla u, \nabla \phi_h) &= (f, \phi_h)\\ (\nabla u_h, \nabla \phi_h) &= (f, \phi_h)\end{aligned} \quad \Rightarrow \quad (\nabla(u-u_h), \nabla\phi_h) = 0 \quad \forall \phi_h \in V_h$$

  The approximation error $u - u_h$ is othorgonal on the discrete function space V_h in the H^1-*scalar product*

$$(\nabla u, \nabla v) = \int_\Omega \nabla u \cdot \nabla \phi \, dx.$$

Lemma 4 (Best approximation) *For the finite element error $u - u_h$ it holds*

$$\|\nabla(u-u_h)\| \leq \min_{v_h \in V_h} \|\nabla(u - v_h)\|.$$

- The problem of solving a partial differential equation is reduced to finding a good function space V_h suitable for approximating the solution u
- The Galerkin method will automatically find the "best solution" in V_h

$$\|\nabla(u-u_n)\|^2 = (\nabla(u-u_n), \nabla(u-u_n))$$

$$+ \underbrace{(\nabla(u-u_n), \nabla\varphi_n)}_{=0 \ \forall \varphi_n}$$

$$= (\nabla(u-u_n), \nabla(u-u_n+\varphi_n)) \quad \forall \varphi_n$$

$$(\varphi_n = v_n - u_n \qquad \forall v_n \in V_n)$$

$$= (\nabla(u-u_n), \nabla(u-v_n)) \qquad \forall v_n$$

$$\leq \|\nabla(u-u_n)\| \cdot \|\nabla(u-v_n)\|$$

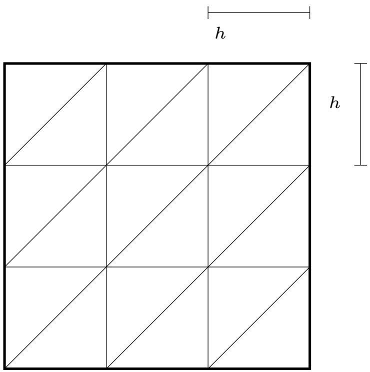
First step

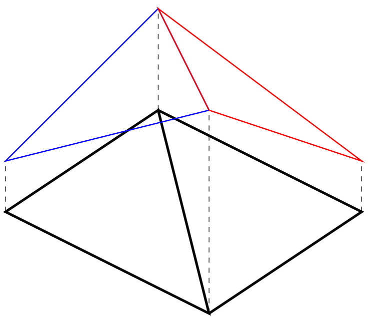
- We discretize the domain Ω by a mesh

- A mesh usually is a set of *elements* which are usually triangles or quadrilaterals in 2d and tetrahedra, prisms, hexahedras in 3d

- We denote elements by $T \in \Omega_h$. All elements together cover the domain Ω, two elements do not overlap

$$\bar{\Omega} = \bigcup_{T \in \Omega_h} \bar{T} \text{ and } T \cap S = \emptyset \quad \forall T \neq S \in \Omega_h.$$





- Triangulation of the square $\Omega = (0,1)^2$ into uniform triangles
- On every triangle define a piece-wise linear function

$$\phi_h^T := \phi_h\big|_T \in \operatorname{span}\{1, x, y\}$$

- If T really is a triangle (not 3 points x_1^T, x_2^T, x_3^T in a line) we can uniquely interpolate three value

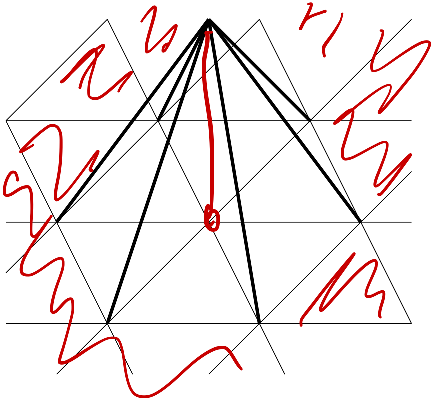
$$\phi_h^T(x_i) = y_i \quad i = 1,2,3$$

- On each edge $e \in \partial T \cap \partial S$ between two triangles, two functions interpolating the same values coincide

$$\phi_h^T = \phi_h^S \text{ on } e = \partial T \cap \partial S.$$

- Therefore: a global function $\phi_h \in V_h$ with $\phi_h|_T \in P^1(T)$ is continuous. This is enough to show

$$\phi_h \in H_0^1(\Omega)$$



We define the basis function $\phi_h^i(x)$ with the following rule:

- ϕ_h^i is linear on each triangle
- It holds $\phi_h^i(x_i) = 1$ and $\phi_h^i(x_j) = 0$ in all other grid points

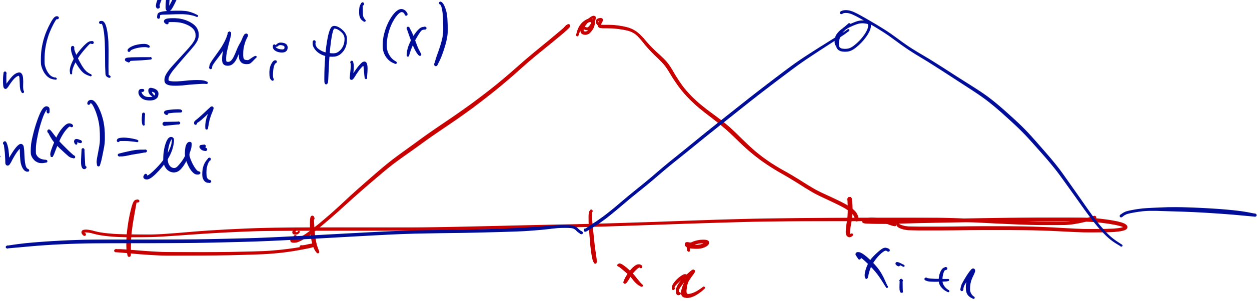
- Called the *linear nodal basis* or *linear Lagrange basis*
- The basis functions have the property $\phi_h^i(x_j) = \delta_{ij}$
- The support of each function is very small

$$\mathrm{supp}(\phi_h^i) = \bigcup_{x_i \in \bar{T},\, T \in \Omega_h} T$$

... On extra slide: linear finite elements on a uniform mesh

$u_h(x) = \sum_{i=0}^{N} u_i \varphi_h^i(x)$

$u_h(x_i) = u_i$



- Finite elements of degree r

$$\phi_h\Big|_T \in P^r(T)$$

Combined to global continuous functions on the complete domain

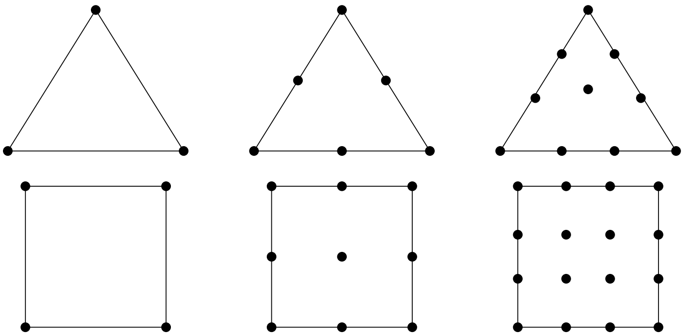
- Finite elements on quadralaterials (bi-linear functions)

$$\phi_h\Big|_T \in Q^1 = \text{span}\{1, x, y, xy\}, \quad \phi_h\Big|_T \in Q^r = \text{span}\{x^i y^j,\ 0 \le i, j \le r\}$$

$\{1, x, y, x^2, y^2, xy\}$

- Everything in 3d...

0



P_1/Q_1 P_2/Q_2 P_3/Q_3

Lemma 5 *Let Ω_h be a finite element mesh of Ω with mesh size h. Let $V_h^{(r)}$ be a finite element space of local degree r. Let $u \in C^{r+1}(\Omega)$. For the finite element interpolation*

$$\boxed{I_h u \in V_h^{(r)}, \quad I_h u(x_i) = u(x_i) \quad \forall x_i \in \Omega_h}$$

it holds

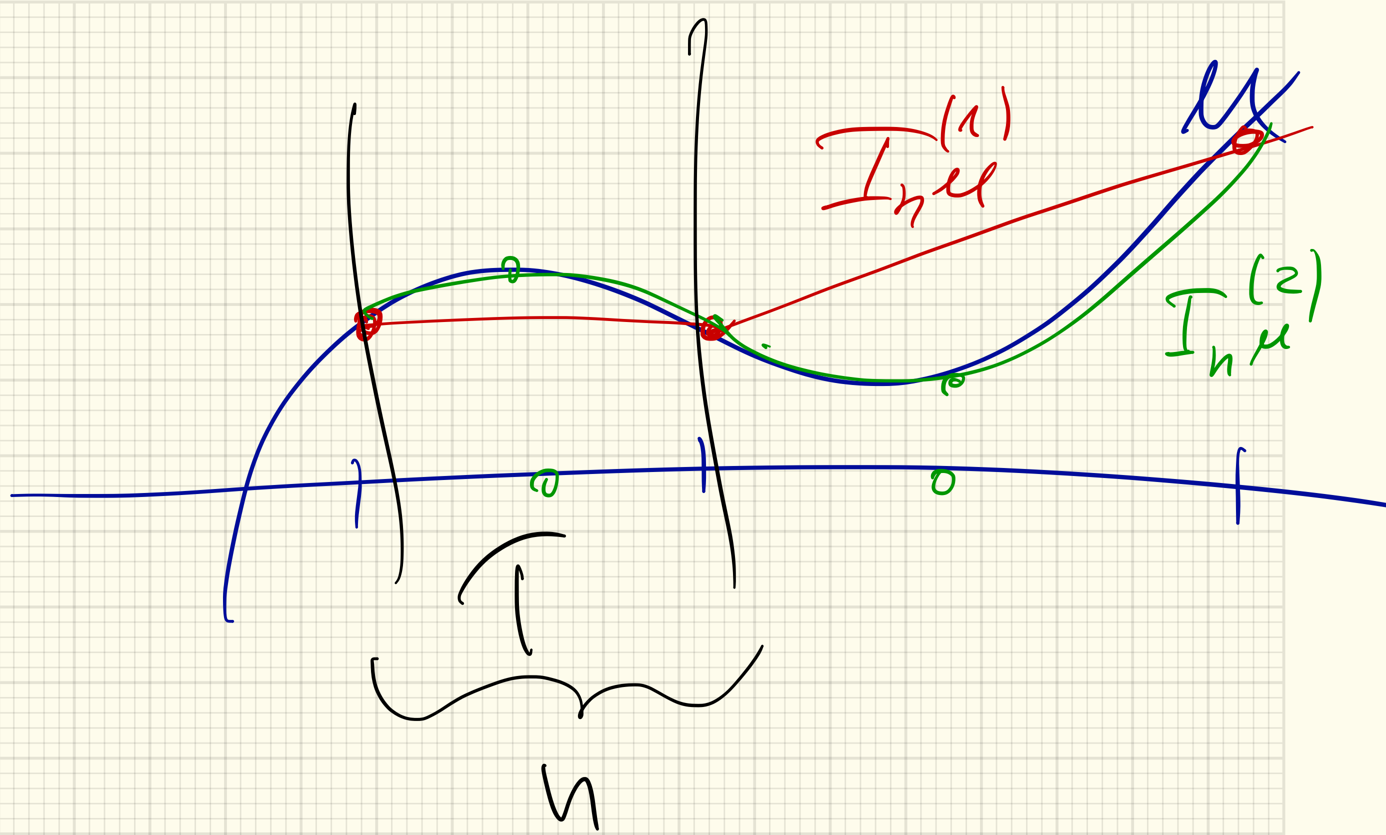
$$\|u - I_h u\| \le c h^{r+1} \|\nabla^{r+1} u\|$$
$$\|\nabla(u - I_h u)\| \le c h^{r} \|\nabla^{r+1} u\|$$
$$\|u - I_h u\|_{L^\infty(\Omega)} \le c h^{r+1} \|\nabla^{r+1} u\|_{L^\infty(\Omega)}$$

- These are *interpolation estimates.*
- They only deal with the approximation quality of a discrete function space $V_h^{(r)} \subset V$
- They do not say anything about a partial differential equation

Best-Approx: $\|\nabla(u-u_h)\| \le \|\nabla(u-v_h)\|$ $\quad v_h \in V_h$

$\le \|\nabla(u - I_h u)\|$

lin FE $\quad \le ch\|\nabla^2 u\|$



$I_h^{(1)} u$
u
$I_h^{(2)} u$
T
h

Lemma 6 *Let Ω_h be a finite element mesh of Ω with mesh size h. Let $V_h^{(r)}$ be a finite element space of local degree r. Let $u \in C^{r+1}(\Omega)$ be the solution to the Laplace equation For the finite element solution*

$$u_h \in V_h \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h$$

it holds

$$\|u - u_h\| \leq ch^{r+1} \|\nabla^{r+1} u\|$$
$$\boxed{\|\nabla(u - u_h)\| \leq ch^r \|\nabla^{r+1} u\|}$$

Energy-Norm estimate

and

$$\|u - I_h u\|_{L^\infty(\Omega)} \leq ch^2 \log(h) \|\nabla^2 u\|_{L^\infty(\Omega)}$$

for linear finite elements.

Follows from Best-Approx. & Interpolation

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Summary

- We start with the variational formulation

$$u \in V = H_0^1(\Omega) \quad (\nabla u, \nabla \phi) = (f, \phi) \quad \forall \phi \in V$$

- We construct a mesh Ω_h of Ω and a finite element space

$$V_h \subset V$$

- The finite element solution is given as

$$u_h \in V_h \subset V \quad (\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h$$

- The finite element space has a finite basis

$$V_h = \text{span}\{\phi_h^1, \dots, \phi_h^N\}, \quad \dim(V_h) = N.$$

$$u_h(x) = \sum_{i=1}^{N} u_i \varphi_h^i(x)$$

$$u_i \in \mathbb{R}$$

- We write the unknown solution as

$$u_h(x) = \sum_{j=1}^{N} u_j \phi_h^j(x), \quad u_1, \dots, u_N \in \mathbb{R}$$

$u \in \mathbb{R}^N$

- As the scalar product is linear it holds

$$(\nabla u_h, \nabla \phi_h) = \sum_{j=1}^{N} (\nabla \phi_h^j, \nabla \phi_h) u_j$$

and thus

$$(\nabla u_h, \nabla \phi_h) = (f, \phi_h) \quad \forall \phi_h \in V_h \quad \Leftrightarrow \quad \sum_{j=1}^{N} (\nabla \phi_h^j, \nabla \phi_h^i) u_j = (f, \phi_h^i) \quad \forall i = 1, \dots, N.$$

- This is equivalent to

$$Au = b$$

with the matrix A and right hand side vector b given as

$$A \in \mathbb{R}^{N \times N}, \quad b \in \mathbb{R}^N, \quad A_{ij} = (\nabla \phi_h^j, \nabla \phi_h^i), \quad b_i = (f, \phi_h^i).$$

The finite element matrix

- is symmetric

$$A_{ij} = (\nabla \phi_h^j, \nabla \phi_h^i) = (\nabla \phi_h^i, \nabla \phi_h^j) = A_{ji}$$

- is positive definite

$$\langle Au, u\rangle = \sum_{ij} A_{ij} u_j u_i = \sum_{ij} (\nabla \phi_h^j, \nabla \phi_h^i) u_j u_i = \sum_{ij} (u_j \nabla \phi_h^j, u_i \nabla \phi_h^i) = (\nabla u, \nabla u) = \|\nabla u\|^2 > 0$$

- is **sparse**

$$A_{ij} = (\nabla \phi_h^j, \nabla \phi_h^i) = \int_\Omega \nabla \phi_h^j \cdot \nabla \phi_h^i \, dx = 0 \text{ if } \operatorname{supp}(\phi_h^j) \cap \operatorname{supp}(\phi_h^i) = \emptyset$$

  Which means

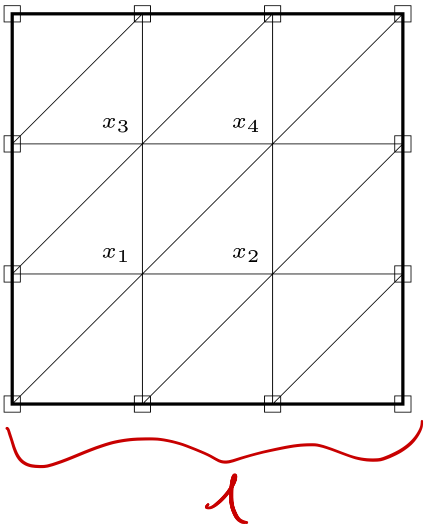
$$A_{ij} \neq 0 \text{ only possible, if } \exists T \in \Omega_h : \; x_i, x_j \in \bar{T}$$

- We call the non-zero indices the **sparsity pattern**

$$S(A) = \{(i,j) \in \{1, \dots, N\}^2 \; : \; A_{ij} \neq 0\}$$

  It holds $\#S(A) = \mathcal{O}(N)$

- All finite element matrices for the Laplace problem have these properties
- The specific form (sparsity pattern and value of entries) depends on
  - The mesh and the mesh size h
  - The finite element degree r
  - The dimension of the problem
  - The numbering of the unknowns (called **degrees of freedom, dof's**) in the mesh



- A possible way to number the unknowns
- Here we have 4 inner points and 12 boundary points where we know that the solution is always zero

$$-\Delta u = f \text{ in } \Omega = (0,1)^d$$
$$u = 0 \text{ on } \partial\Omega$$

Lexicographic sorting

- We only store **inner points** not the one on the boundary. Here it holds $u = 0$
- First x-index, then y, then z (in 3d)
- On a uniform mesh of the square or cube $\Omega = (0,1)^d$ with mesh size $h = 1/M$ and $M-1$ inner nodes in every direction, i.e. $N = (M-1)^d$ overall nodes this gives

$$\mathbf{x}_i = (x_k, x_l) \quad i = (M-1)l + k, \quad \mathbf{x}_i = (x_k, x_l, x_m) \quad i = (M-1)^2 m + (M-1)l + k$$

Model matrix linear finite elements on a uniform triangular mesh

$A \in \mathbb{R}^{N \times N}$

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & -I & 0 & \cdots & 0 \\ -I & \mathbf{B} & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & \mathbf{B} & -I \\ 0 & \cdots & 0 & -I & \mathbf{B} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 4 & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix}. \in \mathbb{R}^{M-1 \times M-1}$$

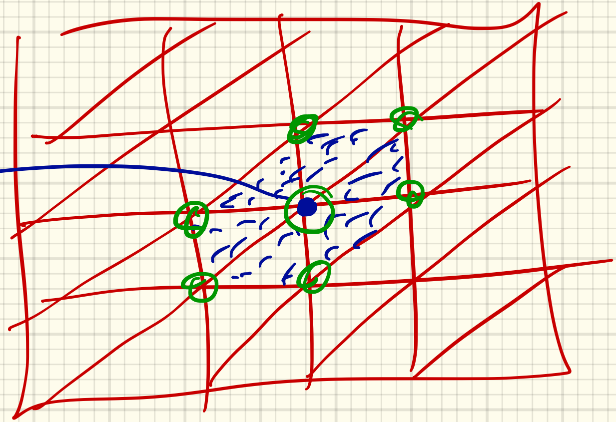
... derivation on extra slide

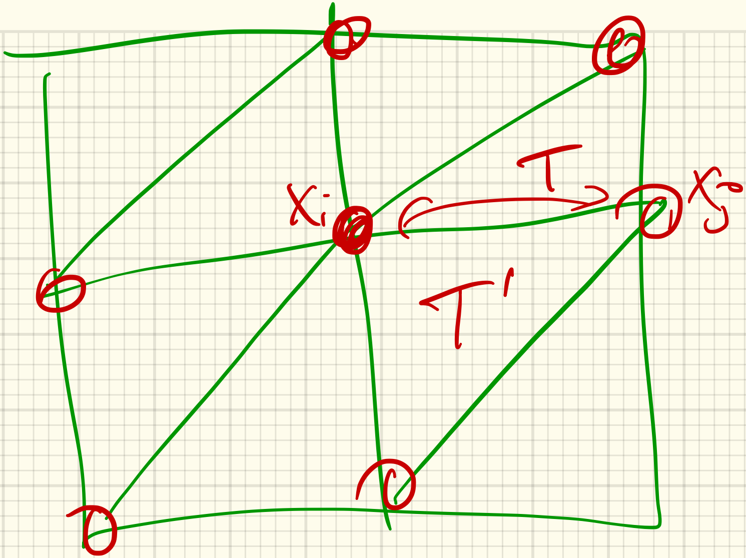
$I \in \mathbb{R}^{(M-1) \times (M-1)}$

$$A_{ij} = (\nabla \varphi^j, \nabla \varphi^i) = \int_\Omega \nabla \varphi^j \cdot \nabla \varphi^i \, dx$$

$$= \sum_{\substack{T \in \Omega_h \\ x_i \in \overline{T} \text{ and} \\ x_j \in \overline{T}}} \int_T \nabla \varphi^j \cdot \nabla \varphi^i \, dx$$

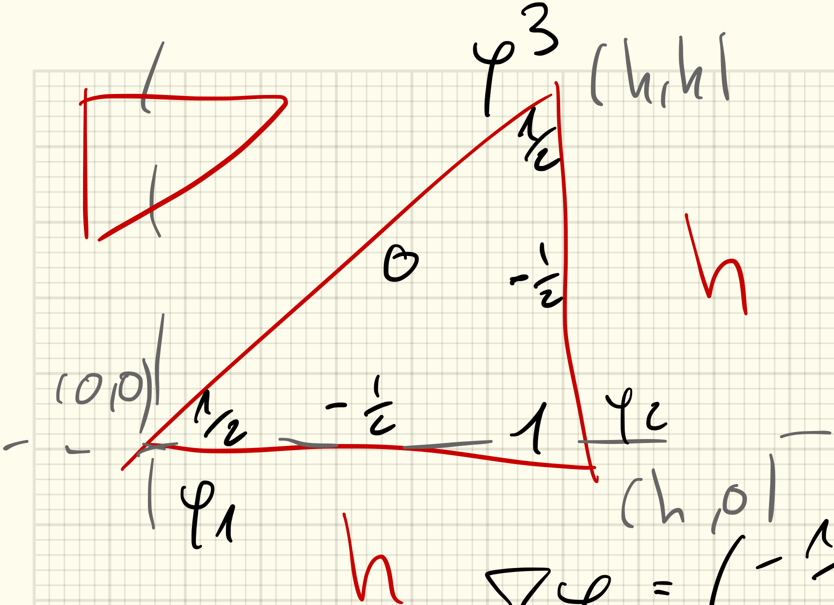
x_i





$$\int_T \nabla\varphi^i \cdot \nabla\varphi^j \, dx$$

$$A_{ij} = \int_T \nabla\varphi^j \cdot \nabla\varphi^i \, dx + \int_{T'} \nabla\varphi^i \cdot \nabla\varphi^j \, dx$$



$$\varphi_1(x,y) = 1 - \frac{x}{h}$$

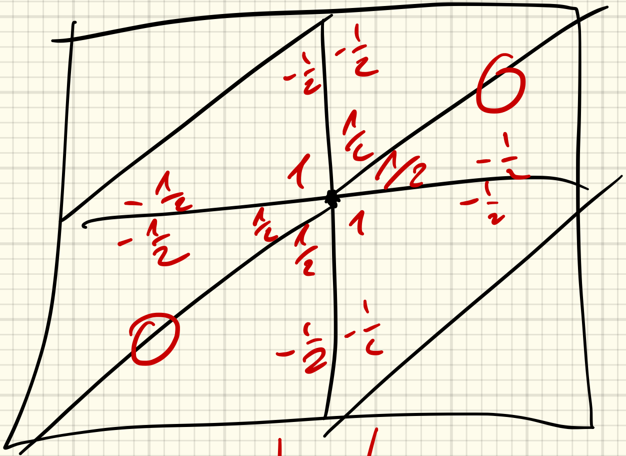
$$\varphi_2(x,y) = \frac{x-y}{h}$$

$$\varphi_3(x,y) = \frac{y}{h}$$

$$\nabla\varphi_1 = \begin{pmatrix} -1/h \\ 0 \end{pmatrix}, \quad \nabla\varphi_2 = \begin{pmatrix} 1/h \\ -1/h \end{pmatrix}$$

$$\nabla\varphi_3 = \begin{pmatrix} 0 \\ 1/h \end{pmatrix}$$

$$\int_T \nabla\varphi^j \cdot \nabla\varphi^i \, dx = \underbrace{|T|}_{\frac{1}{2}h^2} \nabla\varphi^j \cdot \nabla\varphi^i$$



$$A_{ii} = 4$$

$$A_{i,i\pm 1} = -1$$

$$A_{i,i\pm(n-1)} = -1$$

Stencil-Notation

$$S_n = \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}$$

- The matrix looks the same in every inner point of the mesh
- In 2d: we always have 4 on the diagonal and off-diagonal couplings to each side where the entry is -1
- We introduce the **stencil-notation**

$$S_{2d} = \begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}, \quad S_{3d} = h \quad \begin{matrix} & & & -1 \\ & \begin{bmatrix} & -1 & \iddots \\ -1 & 6 & -1 \\ \iddots & -1 & \end{bmatrix} & & \\ -1 & & & \end{matrix}$$

- On a uniform mesh (mesh size exactly h in every direction) the stencil notation does not depend on the numbering of unknowns

$$\mathbf{A} = \begin{pmatrix} \mathbf{B} & -I & 0 & \cdots & 0 \\ -I & \mathbf{B} & -I & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -I & \mathbf{B} & -I \\ 0 & \cdots & 0 & -I & \mathbf{B} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 4 & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix}.$$

- Symmetric, positive definite (regular) as all finite element matrices
- It is very sparse. Only 5 entries (in 2d) and 7 entries (in 3d) in a row

$$A_{i,i} \neq 0, \quad A_{i,i\pm 1} \neq 0, \quad A_{i,i\pm(M-1)} \neq 0 \text{ and in 3d} \quad A_{i,i\pm(M-1)^2} \neq 0$$

- It is (this depends on the sorting of the dof's) a **band-matrix** such that

$$A_{ij} = 0 \quad |i-j| > N^{\frac{1}{2}} \text{ in 2d and } A_{ij} = 0 \quad |i-j| > N^{\frac{2}{3}} \text{ in 3d}$$

- The right hand side vector $b \in \mathbb{R}^N$ is defined as

$$b_i = (f, \phi_h^i) = \int_\Omega f \phi_h^i \, dx$$

- Since f is an arbitrary function we integrate f numerically
- One example (suitable for linear finite elements) is the trapecoidal rule

$$b_i = \sum_{T \in \Omega_h} \sum_{j=1}^{3} \frac{|T|}{3} f(x_j^T) \cdot \phi_h^i(x_j^T) \text{ where } \phi_h^i(x_j^T) \neq 0 \text{ only, if } x_j^T = x_i$$

- On uniform meshes it holds

$$b_i = h^d f(x_i) \text{ in 2d and 3d}$$

- The finite element problem is equivalent to solving a linear system of equations

$$Ax = b$$

- The linear system $Ax = b$ can be easily solved by Gaussian elimination in

$$\mathcal{O}\left(N \cdot B^2\right)$$

operations (B is the band-width)

- This is

$$O_{2d} = \mathcal{O}\left(N^2\right), \quad O_{3d} = \mathcal{O}\left(N^{2+\frac{1}{3}}\right)$$

$\mathcal{O}(N)$

- We simulate the mechanical behavior of a tabletop of dimension $2m \times 1m \times 2cm$. The finite element mesh consists of small boxes of size
$$h = 1cm$$
$\frac{1}{2}$ cm

  Hence, we need
$$N = \frac{200cm}{1cm} \cdot \frac{100cm}{1cm} \cdot \frac{2cm}{1cm} = 40\,000$$
320 000

  The effort for solving the problem is
$$O_{3d} = 40\,000^{2+\frac{1}{3}} \approx 10^{10} \text{ Operations}$$
$\approx 10^{12}$ Op.

  On a modern computer this takes a few seconds.

- If we approximate the thin plate with a 2d-model the number of unknowns reduces to
$$N = \frac{200cm}{1cm} \cdot \frac{100cm}{1cm} = 20\,000$$
$\frac{1}{2}$ 80.000

  and the effort is reduced to
$$O_{2d} = 20\,000^2 \approx 10^8 \text{ Operations.}$$
10^9

  This problem can be solved in less than a second

- Compute the flow around an aircraft of length $L = 100m$. The vortices are very small $< 1mm$. We need a mesh size

$$h \ll 1mm = 1/1000m$$

To compute on a domain of $500m \times 100m \times 100m$ we need

$$N = \frac{500}{1/1000} \cdot \frac{100}{1/1000} \cdot \frac{100}{1/1000} > 10^{15} \text{ unknowns}$$

and therefore

$$O_{3d} > 10^{35} \text{ Operations}$$

Which - on a non-existing - exaflop-Computer (10^{18} Operations per second) - takes

$$10^{12} \text{ years}$$

thanks!